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Inverse scattering method applied to degenerate two-photon propagation in the low-excitation limit

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Abstract. The propagation of an optical pulse in a two-photon absorber is studied under the conditions that the initial pulse is detuned from exact resonance, and the population of the upper level remains small. The problem is solved by use of a recently developed variant of the inverse scattering technique on a finite interval.

1. Introduction

Degenerate two-photon propagation (DTPP) means the coherent propagation of an optical wave through a system of two-level atoms under the condition that two-photon absorption and emission processes occur. The atomic energy difference, ΔE, and the photon frequency, ω_0, then fulfil $\Delta E = 2\hbar\omega_0$. This system is the limit of (nondegenerate) TPP where there are photons of two different frequencies (ω_1 and ω_2), propagating in a two-level atomic system fulfilling $\Delta E = \hbar(\omega_1 + \omega_2)$. The latter is closely related to *stimulated Raman scattering*, both physically and mathematically, which is characterized by $\Delta E = \hbar(\omega_1 - \omega_2)$. All three systems are IST integrable in principle; cf [1,2]. The DTPP system, however, is unique among integrable systems, in that standard methods have not yielded a solution for this system. Thus instead of attempting to solve the full complex problem, we shall now take the approach of looking at various limits of these equations, in order to better understand, and gain insight into, the more general system. In the process, we have found that various limits are related to other important systems. Here, by treating the low-excitation limit of the DTPP equations, we find that these equations become the same as the equations of second harmonic generation (SHG) with walk-off ($v_1 \neq v_2$) and with phase modulations (chirps) of the pulse envelope. Interestingly enough, both of these systems are of current nonlinear optical interest. SHG provides a very simple means for generating pulses at frequencies either twice, or half that of the pump. Thus solutions of one would also apply to the other. Here, we will study a simple model initial-boundary-value problem from DTPP by use of the SHG equations, and also present a new method for constructing solutions of these nonsoliton, finite-interval problems.

Earlier work on DTPP with amplitude-modulated pulses include the analytical solutions given by Poluektov *et al* [3], which have been generalized recently [4]. In another paper [5], additional analytical solutions have been constructed by means of Bäcklund transformations.

Furthermore, for the same system, the one-phase periodic solution has been constructed, and its modulation has been investigated by the Whitham method [6].

Here we look at DTPP in the SHG limit when one is off-resonance, and when the system is on a finite or semi-infinite interval. We study the low-excitation limit of this system for two reasons. First, the physically interesting regime certainly includes the low-excitation limit, since one seldom has enough photon flux to cause the full excitation of all atoms or molecules in a medium. Second, this limit avoids many of the complications of the full problem, which is still unsolved. As one example, in the low-excitation limit, as we shall see below, although the Lax evolution operator is singular for all values of the eigenvalue, it is still in a form where the evolution of the scattering data can be readily determined. However, for the general problem, this evolution operator has a different singular structure, one which has a much more complex (and rich) evolution of the scattering data, the evolution of which has yet to be detailed.

In solving this SHG limit of DTPP, we have found a new and uniquely different method of solution for these nonsoliton, but integrable systems where, in the Ablowitz–Kaup–Newell–Segur (AKNS) notation [11], $r = +q^*$. For these $r = +q^*$ AKNS systems, when the potentials vanish at infinity, or are on compact support, no bounded soliton solutions exist. The only nonsingular and regular solutions which exist are those nonlinear solutions called 'radiation', which are essentially nonlinear plane waves, and are represented by the continuous spectra of the scattering problem of the Lax pair. However, in the case of a semi-infinite or a finite interval, one does not require the entire continuous spectra in order to determine the solution. One can get by with only a countable amount of the scattering data. We demonstrate how, for this and other similar hyperbolic $r = +q^*$ systems, one can represent the scattering data by an infinity of poles in the *lower* half complex plane, and their residues. We will call these poles, 'virtual solitons'. We then demonstrate that one can solve the Gelfand–Levitan–Marchenko (GLM) integral equations in terms of these virtual solitons. Such a solution will be an infinite sum. However, it can be obtained numerically, and we do verify that the sum over the virtual solitons, in the lower half complex plane, do give the correct solution. Furthermore, in the limit of vanishing detuning, we verify that we can reconstruct the known exact solution of this C-integrable problem [10]. We also verify that, as the number of poles are increased, the pole solution does approach the numerical solution of the partial differential equations (PDE), (11). The limit on the number of poles used is simply limited by the power of the computer used, and by the power of the symbolic and numerical software used.

In section 2, we show that the low-excitation limit of DTPP equations is the same as the SHG equations, and we then give the Lax pair. In section 3, we discuss the initial-boundary-value problem that we will solve, and then proceed to reduce it to a canonical form. In section 4, we review the recent reformulation of the inverse scattering transform (IST) for a finite or semi-infinite interval [10], emphasizing how, for these hyperbolic-type systems, one can solve for the evolution of the scattering data by means of an 'effective' scattering matrix. We then proceed to show how the integral over the scattering data, in the GLM equations, can be reduced to an infinite sum over the residues of the poles in the lower half complex plane. We emphasize here and later that this method will always be restricted to the finite or semi-infinite interval, since for an infinite interval problem, certain integrals could not then exist.

In section 5, we discuss the simple one-pole potential for a single virtual soliton, and discuss its possible forms, one of which is periodic and singular. However, for the problem at hand, we find that this form never occurs, and that the solution is regular and nonsingular. We also demonstrate that this solution is valid in the limit of small detuning, and that this limit recovers the C-integrable real amplitude solution case. In section 6, we treat the general N-pole approximation. In section 7, we compare the solution found in the previous section, against the numerical solution obtained by direct numerical integration of the PDEs, (11). We find excellent agreement, with the N-pole approximation converging well toward the numerical solution. Summary and conclusions are given in section 8.

2. The low-excitation limit of DTPP is equivalent to SHG

Let us start with the DTPP equations in the form

$$\partial_\tau r_- = -2\mathrm{i}(\mathcal{E}^2 r_3 + g\mathcal{E}\mathcal{E}^* r_-) \tag{1}$$

$$\partial_\tau r_3 = \mathrm{i}(r_+\mathcal{E}^2 - r_-\mathcal{E}^{*2}) \tag{2}$$

$$\partial_\chi \mathcal{E} = \frac{\mathrm{i}}{2}(r_-\mathcal{E}^* - g r_3 \mathcal{E}) \tag{3}$$

where $\mathcal{E}$ is the slowly varying electric field amplitude. The Bloch vector (r_1, r_2, r_3), $r_\pm \equiv r_1 \pm \mathrm{i} r_2$ fulfils

$$r_1^2 + r_2^2 + r_3^2 = 1. \tag{4}$$

Here χ and τ are space and retarded time variables respectively. They are related to the laboratory space and time coordinates x, t by $\chi = x$, $\tau = t - x/v$, with v being the group velocity, and the units are chosen to make the coupling coefficients unity. g is a constant that has two different aspects [4]. In (1) it describes the dynamic Stark shift. In (3), it describes a population dependent refractive index. Why these two different aspects have the same coefficient, and the same numerical value, is not fully understood [4].

These equations only differ from what was presented in former papers [4,5] by the omission of the S-vector notation. In this paper, we shall go directly to the low-excitation limit of these equations, and treat that problem.

To obtain the low-excitation limit, all that we have to do, is to take $r_+r_- \ll 1$, $r_3 \simeq -1 + r_+r_-/2$, but in (1), we replace r_3 by simply -1. Then the above system reduces to

$$\partial_\tau r_- = 2\mathrm{i}(\mathcal{E}^2 + g\mathcal{E}\mathcal{E}^* r_-) \tag{5}$$

$$\left(\partial_\chi - \frac{\mathrm{i}g}{2}\right)\mathcal{E} = \frac{\mathrm{i}}{2}\left(r_-\mathcal{E}^* - \frac{1}{2} g r_+ r_- \mathcal{E}\right). \tag{6}$$

To reduce this system to the SHG equations, we perform a rather simple phase transformation, which will transform g to zero. From the conservation law

$$\partial_\tau(r_+r_-) + 4\partial_\chi(\mathcal{E}\mathcal{E}^*) = 0 \tag{7}$$

it follows that there exists a function $\delta(\chi, \tau)$, such that

$$\partial_\chi \delta = \frac{g}{4} r_+ r_- \qquad \partial_\tau \delta = -g\mathcal{E}\mathcal{E}^*. \tag{8}$$

The phase transformation is

$$\tilde{\mathcal{E}} = \mathcal{E}\mathrm{e}^{\mathrm{i}(\delta - g\chi/2)} \qquad \tilde{r}_- = r_-\mathrm{e}^{\mathrm{i}(2\delta - g\chi)} \tag{9}$$

from which we see that $\tilde{\mathcal{E}}$, $\tilde{r}_-$ satisfy exactly the same equations as (5), (6) but now, $g = 0$. Next, define

$$q_2 = -\mathrm{i}\tilde{r}_- \qquad q_1 = \tilde{\mathcal{E}} \qquad \tilde{\chi} = \chi/2 \tag{10}$$

and then, upon omitting the tilde on the χ, we arrive at the equations

$$\partial_\chi q_1 = -2 q_2 q_1^* \qquad \partial_\tau q_2 = q_1^2 \tag{11}$$

which are the equations for SHG, see, e.g. [7–9]. The scaling of the variables have been chosen such that the conservation law (7) becomes $\partial_\chi(q_1 q_1^*) + 2\partial_\tau(q_2 q_2^*) = 0$ wherein $q_k q_k^*$ can be interpreted as photon current densities, up to a common constant factor. In the DTPP problem, the characteristic coordinate χ coincides with the laboratory coordinate x. For the SHG problem, when both fields are optical fields, the connection between the characteristic coordinates χ, τ and the laboratory coordinates, x, t is

$$x = \chi + \tau \qquad t = \chi/v_1 + \tau/v_2 \tag{12}$$

so that the derivatives are transformed according to

$$\partial_\chi = \partial_x + \frac{1}{v_1}\partial_t \qquad \partial_\tau = \partial_x + \frac{1}{v_2}\partial_t \tag{13}$$

where v_1, v_2 denote the group velocities of the optical fields.

Once we have the equations in the form of (11), they can be shown to be the integrability conditions for the Lax pair [7]

$$\partial_\chi \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = U \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \equiv \begin{pmatrix} -\mathrm{i}\zeta & 2q_2 \\ 2q_2^* & \mathrm{i}\zeta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{14}$$

$$\partial_\tau \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = V \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \equiv \frac{\mathrm{i}}{\zeta} \begin{pmatrix} -q_1^* q_1 & -q_1^2 \\ q_1^{*2} & q_1^* q_1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{15}$$

These equations have been studied in [7]. There, the scattering data are defined, and it was shown that the evolution of the scattering data would satisfy a second-order ordinary differential equation (ODE). Here, we shall take a model initial-boundary-value problem, and demonstrate the solution of it by a new technique, using ‘virtual solitons’.

3. Formulation of the problem

The typical physical problem to be solved for DTPP (here restricted to the low excitation limit) is that of an optical pulse $q_1(0, \tau)$ incident on the medium at $\chi = 0$. At $\tau = \tau_0$, the medium is prepared in a given atomic state $q_2(\chi, \tau_0)$, where τ_0 could be finite. In particular, it could be prepared at $\tau = 0$, or even at $\tau \to -\infty$. With these conditions, this is a *Goursat problem*, i.e., where the initial values are prescribed along the characteristics. If the atoms were initially in the ground state, then $q_2(\chi, 0) \equiv 0$. This problem, we will call the *restricted* Goursat problem. The only real difference between these two problems is that for the nonrestricted problem, the initial value of the effective scattering matrix would be some nontrivial unitary matrix, and not the unit matrix.

Now, in the case of SHG, the typical physical problem to be solved is not a Goursat problem. For a SHG initial-boundary-value problem, typically the fields q_1, q_2 both are given at $x = 0$, which in this case, is not a characteristic, cf (12). Such a problem is called a *Cauchy problem* and is more complex than the restricted Goursat problem considered here. Nevertheless the solution of this Goursat problem will give insight into understanding the Cauchy problem. (For the solution of the Cauchy problem when restricted to purely amplitude-modulated pulses see [9].)

Let us return and study the restricted Goursat problem for DTPP. The simplest such problem would be one where q_1 has a constant amplitude and has a finite width. That case has been treated in [9]. Here we will consider the case when a phase variation is present, which cannot be solved by the techniques in [9]. The simplest phase variation is a nonzero linear variation, which corresponds to an unchirped, but off-resonance pulse. Thus we shall take

$$q_1(0, \tau) = q_{10}\mathrm{e}^{\mathrm{i}\Delta\tau} \qquad q_2(\chi, 0) = 0 \tag{16}$$

for $0 < \tau < T_0$, and zero outside, and where T_0 is the width, q_{10} is a constant, and Δ is the nonzero frequency mismatch, which is constant and real. The physical interpretation of this pulse is that the incident wave is a square pulse, with a constant frequency mismatch tuned away from the exact resonance. We will now show that, due to the symmetry of these equations, we may set both parameters in (16) equal to unity.

Symmetry. Because there is an arbitrary constant phase, we may take q_{10} as real and positive. Since the SHG equations (11) have only real coefficients, they are invariant under the transformation $(q_1, q_2) \to (q_1^*, q_2^*)$. Therefore we may take $\Delta > 0$. Furthermore, it is easy to see that these equations are invariant as well under the similarity transformation

$$\tilde{\chi} = \frac{q_{10}^2}{\Delta}\chi \qquad \tilde{\tau} = \Delta\tau \qquad \tilde{q_1} = \frac{q_1}{q_{10}} \qquad \tilde{q_2} = \frac{\Delta}{q_{10}^2}q_2. \tag{17}$$

As a result of these transformations, without loss of generality, we may therefore take

$$q_{10} = \Delta = 1 \tag{18}$$

for $0 < \tilde{\tau} < T_0\Delta$, and zero otherwise. This is the restricted Goursat problem (16) that we shall solve. Now, let us consider how we may use the above Lax pair to solve this problem.

4. The inverse scattering scheme

We take (14) as the *scattering problem* with $q_2(\chi, \tau)$, with τ fixed, as the *potential*. This problem is the same as the Zakharov–Shabat (ZS) problem, with $r = +q^*$. This problem is the self-adjoint form, and thus, on the infinite interval, for potentials vanishing as $|\chi| \to \infty$, there are no bound state eigenvalues: namely, for bounded eigenfunctions, ζ can be real only. The other half of the Lax pair, (15), will then be used to determine the τ-evolution of scattering data.

4.1. The direct scattering problem

This, of course, is trivial. In the DTPP problem, the potential at $\tau = 0$ corresponds to the initial molecular excitation level, which we have taken to vanish. Thus the initial S-matrix is the unit matrix for any finite interval, as well as for the full-line or for the half-line.

4.2. The τ-evolution of the effective S-matrix

As we have indicated before, one may solve this problem by the use of an effective S-matrix. How one may do this has been extensively treated in the foregoing paper [10]. What was shown there was the following: principally, in order to obtain the τ-evolution of the S-matrix in a finite interval, $0 < \chi < \chi_f$, one needs both $V(0, \tau)$—which is known from the initial data—and also $V(\chi_f, \tau)$—which cannot be readily obtained until the solution is obtained, and therefore is not readily known. However, as discussed in [10], and as is well known, to construct the solution by the method of characteristics, one never needs to know the solution at $\chi = \chi_f$, which is into the 'future'. Thus the kernels in the GLM equations must be independent of $V(\chi_f, \tau)$, and only dependent on $V(0, \tau)$. In [10], it was then shown that if one defined the evolution of an *effective* S-matrix by simply ignoring $V(\chi_f, \tau)$, then this effective S-matrix did contain all the necessary information for determining the evolution of the scattering data. In fact, it was proven there that for AKNS problems, and for V-matrices vanishing asymptotically for $|\zeta| \to \infty$ (equivalent to being a hyperbolic-type equation), the effective S-matrix gives correct physical results *inside the considered interval*. All these conditions are fulfilled for our present problem.

Once we have established this, then according to [10], the τ-evolution of the *effective* S-matrix is given by

$$\partial_\tau \boldsymbol{S}^{\mathrm{eff}} = -\boldsymbol{S}^{\mathrm{eff}} V(0,\tau) \qquad \boldsymbol{S}^{\mathrm{eff}}(0) = \mathbf{1} \tag{19}$$

with

$$V(0,\tau) = \frac{\mathrm{i}}{\zeta}\begin{pmatrix} -1 & -\mathrm{e}^{2\mathrm{i}\tau} \\ \mathrm{e}^{-2\mathrm{i}\tau} & 1 \end{pmatrix}. \tag{20}$$

The solution for the effective S-matrix is readily found to be

$$\boldsymbol{S}^{\mathrm{eff}}(\tau) = (\mathbf{1}\cos(w\tau) + \mathrm{i}w^{-1}M\sin(w\tau))T(\tau) \tag{21}$$

where

$$w = w(\zeta) = \sqrt{2/\zeta + 1} \tag{22}$$

$$M = \zeta^{-1}(\sigma_3 + \mathrm{i}\sigma_2) + \sigma_3 \tag{23}$$

$$T(\tau) = \begin{pmatrix} \exp(\mathrm{i}\tau) & 0 \\ 0 & \exp(-\mathrm{i}\tau) \end{pmatrix} \tag{24}$$

with σ_k being the standard Pauli spin matrices. From the above effective S-matrix, one finds the 'reflection coefficient' in the usual manner, and it is given by

$$c(\zeta,\tau) \equiv -\frac{\boldsymbol{S}^{\mathrm{eff}}_{12}}{\boldsymbol{S}^{\mathrm{eff}}_{11}} = \frac{\mathrm{e}^{2\mathrm{i}\tau}}{\zeta m[w(\zeta),\tau]} \tag{25}$$

where

$$\begin{aligned} m(w,\tau) &\equiv [\mathrm{i}w\cot(w\tau) - (w^2+1)/2] \\ &= \tfrac{1}{2}(w + \mathrm{i}\tan(w\tau/2))(-w + \mathrm{i}\cot(w\tau/2)) \\ &= \frac{[(w+1)\mathrm{e}^{\mathrm{i}w\tau} + (w-1)][(w+1)\mathrm{e}^{\mathrm{i}w\tau} - (w-1)]}{2(1-\mathrm{e}^{\mathrm{i}w\tau})(1+\mathrm{e}^{\mathrm{i}w\tau})}. \end{aligned} \tag{26}$$

As pointed out in [10], although this reflection coefficient is not the full reflection coefficient, nevertheless, in the integral below, (27), it will give the correct value, when x is inside the interval of interest.

4.3. Preparation for the inverse scattering problem: poles and residues

The solution requires that one evaluates the integral given below, in the complex ζ-plane. Since the integrand is an analytic function, this integral can be evaluated in terms of the poles and residues. In this case, it turns out that all the poles are in the lower half complex ζ-plane. Furthermore, one can see that when $z > 0$ and $|\zeta| \to \infty$ in the lower half complex plane, the integrand vanishes exponentially. Similarly, when $z < 0$ and $|\zeta| \to \infty$ in the upper half complex plane, the integrand again vanishes exponentially. Thus this integral can be reduced to only contributions from the poles.

The integral that we have to compute is

$$G(z,\tau) = \frac{1}{2\pi}\int_{\mathcal{C}} c(\zeta)\mathrm{e}^{-\mathrm{i}\zeta z}\,\mathrm{d}\zeta = \frac{\mathrm{e}^{2\mathrm{i}\tau}}{2\pi}\int_{\mathcal{C}} \frac{\mathrm{e}^{-\mathrm{i}\zeta z}}{m[w(\zeta),\tau]}\frac{\mathrm{d}\zeta}{\zeta}. \tag{27}$$

The integration curve $\mathcal{C}$ goes from $-\infty$ to $+\infty$ in the complex ζ-plane, and passes above all poles of the integrand. The location of the poles of c, according to (25), (26), will be determined by

$$\mathrm{e}^{\mathrm{i}w\tau} = \pm\frac{w-1}{w+1}. \tag{28}$$

To solve for the roots of (28), we break w into real and imaginary parts, $w = w_R + \mathrm{i}w_I$, from which we get

$$\mathrm{e}^{-w_I\tau} = \left|\frac{(w_R - 1) + \mathrm{i}w_I}{(w_R + 1) + \mathrm{i}w_I}\right|. \tag{29}$$

For $\tau > 0$ any solution necessarily fulfils

$$w_R w_I \geqslant 0 \tag{30}$$

and then implies that the corresponding image $\zeta = 2/(w^2 - 1)$ cannot be in the upper half ζ-plane.

Because $m(w, \tau)$ is an even function of w we may consider $w_R > 0$ only. The required poles can be expressed by using the reverse functions with respect to $y = -\mathrm{i}z\tan z$ and $y = \mathrm{i}z\cot z$. The respective functions, $z(y)$, are multivalued. Let us define single-valued branches as follows:

$$z = \mathcal{Q}_{2k-1}(y) \qquad y > 0: \qquad y = -\mathrm{i}z\tan z \qquad z \to (k - \tfrac{1}{2})\pi \qquad \text{for} \quad y \to \infty \tag{31}$$

$$z = \mathcal{Q}_{2k}(y) \qquad y > 0: \qquad y = \mathrm{i}z\cot z \qquad z \to k\pi \qquad \text{for} \quad y \to \infty \tag{32}$$

with $k = 1, 2, 3, \ldots$. Then for $0 < y < \infty \mathcal{Q}_j(y)$, $j = 1, 2, 3, \ldots$ moves between $(j-1)\pi/2$ and $j\pi/2$ with $\mathrm{Im}\ \mathcal{Q}_j(y) > 0$. Now

$$w_j = (2/\tau)\mathcal{Q}_j(\tau/2) \tag{33}$$

are the zeros of $m(w, \tau)$ from which the poles for ζ are

$$\zeta_j = \frac{2}{w_j^2 - 1} \qquad j = 1, 2, 3, \ldots. \tag{34}$$

Summarizing, we state that there are no poles on the upper half ζ-plane while there is an infinite number on the lower half ζ-plane, clustering for $w \to \infty$, corresponding to $\zeta = 0$. Remarkably, the movement of these poles as a function of τ can be described by the rather simple differential equation

$$\frac{\mathrm{d}\zeta_j}{\mathrm{d}\tau} = \frac{\zeta_j(2 + \zeta_j)}{\tau + \mathrm{i}\zeta_j}. \tag{35}$$

Now, let us consider the evaluation of the integral in (27). First, when $z < 0$, we simply close the contour from above, and we obtain zero for the integral since there are no poles inside the contour. For $z > 0$, the curve of integration may be closed in the lower half plane, such that it goes around all poles $\zeta_j \quad (j = 1, 2, \ldots)$. The corresponding residues are found to be

$$c_j = \zeta_j^{-1}\mathrm{Res}[m^{-1}[(w(\zeta), \tau)]|_{\zeta=\zeta_j} = \frac{\zeta_j^2(\zeta_j + 2)}{\zeta_j - \mathrm{i}\tau} = \mathrm{i}\zeta_j\frac{\mathrm{d}\zeta_j}{\mathrm{d}\tau}. \tag{36}$$

The function G can be expressed in terms of the residues, as a sum over the poles ζ_j. Note that there is no contribution from the essential singularity at $\zeta = 0$. To see this, consider the first line of (26) and take a half-circle, going up into the upper half w-plane, from $w_n = (n + \frac{1}{2})\pi/\tau$, n integer to, $-w_n$. There, $\cot(w\tau)$ is bounded with some bound, independent of n. Therefore for $n \to \infty$, we have $|m| \to |w_n|^2/2 \propto n^2$. The image of this half-circle in the ζ-plane is a closed curve—approximately a circle—around the origin. This curve crosses over a pass between two singularities, and the corresponding contribution to the integral in (27) decays as n^{-2}. Thus we have

$$G(z, \tau) = -\mathrm{i}\theta(z)\sum_{j=1}^{\infty} c_j \mathrm{e}^{-\mathrm{i}\zeta_j z} \tag{37}$$

where $\theta(z)$ denotes the Heaviside step function.

4.4. The GLM equations

The scattering problem defined by (14) fits into the AKNS scheme [11] with the specification $q = r^* = 2q_2$. Equations (4.39*a*), (4.39*b*) from [11] give the GLM equations for inversion about $x = -\infty$, namely

$$\bar{L}_1(\chi, y) + G(\chi + y) - \int_{-\infty}^{\chi} L_1(\chi, s)G(s + y)\,\mathrm{d}s = 0 \tag{38}$$

$$L_1(\chi, y) + \int_{-\infty}^{\chi} \bar{L}_1(\chi, s)\bar{G}(s + y)\,\mathrm{d}s = 0. \tag{39}$$

In our case ($r = +q^*$) it holds that

$$\bar{G}(z, \tau) = -G^*(z, \tau). \tag{40}$$

The *potential* $q = 2q_2$ then is connected with the solution of the integral equations (38), (39) by

$$q_2(\chi) = q(\chi)/2 = \bar{L}_1(\chi, \chi). \tag{41}$$

It is easy to see that from $G(z) = 0$ for $z < 0$ it follows $q_2(\chi) = 0$ for $\chi < 0$. The substitution of G from (27) into (38), (39) leads to

$$\begin{aligned} &\bar{L}_1(\chi, y) + \mathrm{i}\sum_k c_k \mathrm{e}^{-\mathrm{i}\zeta_k y} \int_{-y}^{\chi} L_1(\chi, s)\mathrm{e}^{-\mathrm{i}\zeta_k s}\,\mathrm{d}s = \mathrm{i}\sum_k c_k \mathrm{e}^{-\mathrm{i}\zeta_k(\chi+y)} \\ &L_1(\chi, y) - \mathrm{i}\sum_k c_k^* \mathrm{e}^{\mathrm{i}\zeta_k^* y} \int_{-y}^{\chi} \bar{L}_1(\chi, s)\mathrm{e}^{\mathrm{i}\zeta_k^* s}\,\mathrm{d}s = 0 \qquad \chi + y > 0. \end{aligned} \tag{42}$$

Obviously, these equations can be solved firstly for a truncated sum with k running from 1 to N, and then taken in the limit of $N \to \infty$. We will shortly illustrate this numerically.

But first, let us take the very simple case of a single pole and work out its solution. This potential will be called a 'virtual soliton', to emphasize that it is not a real soliton, for which the pole would have been in the upper half ζ-plane. We will then show that this potential, in the limit of vanishing detuning, approaches the same as that given in [10].

5. The one-pole potential

Let us assume $\tau \equiv 2y \ll 1$. As per (17), this can be achieved either by the limit of a very short pulse or a vanishing detuning. In either case, we are near the pure amplitude modulation case, considered in [10], which is described by the C-integrable Liouville equation. Under these conditions, we find

$$\begin{aligned} &\mathcal{Q}_1^2 = y\left(\mathrm{i} + \frac{y}{3}\right) + \cdots \qquad && \zeta_1 = -\tau\left(\mathrm{i} + \frac{\tau}{3}\right) + \cdots \\ &\mathcal{Q}_{j+1} = \frac{j\pi}{2} + \frac{2\mathrm{i}y}{j\pi} + \cdots \qquad && \zeta_{j+1} = \frac{2\tau^2}{j^2\pi^2}\left(1 - \frac{4\mathrm{i}\tau}{j^2\pi^2}\right) + \cdots \qquad j = 1, 2, 3, \ldots. \end{aligned} \tag{43}$$

Thus the first pole ζ_1 is well separated from the others, and its residue, according to (36) is

$$c_1 = -\tau(\mathrm{i} + \tau) + \cdots \tag{44}$$

with the higher-order residues vanishing at least like τ^3. Thus a one-pole potential will approximate the solution when the total phase change across the pulse amplitude is small.

Although the general solution is always a sum over the poles, these poles are in the lower half complex ζ-plane. Usually, and particularly when the interval is infinite, only poles in the upper half complex ζ-plane contribute to such integrals. This latter case is very well known, and the solutions for N-poles are called N-soliton solutions. As we see here, when one uses a finite or semi-infinite interval, one can have contributions from poles in the lower half complex ζ-plane. Thus we should analyse the potentials that arise due to these poles, and see what are some of their characteristics. We will call these potentials 'virtual solitons' to distinguish them from usual solitons.

First, let us look at a one-virtual-soliton potential. We take a single pole in the lower half complex plane, located at $\zeta_1 = \xi_1 + \mathrm{i}\eta_1$, $\eta_1 < 0$ with some arbitrary residue c_1. Now, let us construct the corresponding potential

$$\partial_y \begin{pmatrix} \bar{L}_1(\chi, y) \\ L_1(\chi, -y) \end{pmatrix} = M \begin{pmatrix} \bar{L}_1(\chi, y) \\ L_1(\chi, -y) \end{pmatrix} \tag{45}$$

with

$$M = -\mathrm{i} \begin{pmatrix} \zeta_1 & c_1 \\ c_1^* & \zeta_1^* \end{pmatrix}. \tag{46}$$

Equation (45) is solved by

$$\begin{pmatrix} \bar{L}_1(\chi, y) \\ L_1(\chi, -y) \end{pmatrix} = \Phi(\chi + y) \begin{pmatrix} \bar{L}_1(\chi, -\chi) \\ L_1(\chi, \chi) \end{pmatrix} \tag{47}$$

where

$$\Phi(y) \equiv \frac{\mathrm{e}^{-\mathrm{i}\xi_1 y}}{k} \begin{pmatrix} k\cosh(ky) + \eta_1 \sinh(ky) & \mathrm{i}c_1 \sinh(ky) \\ -\mathrm{i}c_1^* \sinh(ky) & k\cosh(ky) - \eta_1 \sinh(ky) \end{pmatrix} \tag{48}$$

with

$$k \equiv \sqrt{\eta_1^2 - c_1 c_1^*}. \tag{49}$$

From

$$\begin{pmatrix} \bar{L}_1(\chi, \chi) \\ L_1(\chi, -\chi) \end{pmatrix} = \Phi(2\chi) \begin{pmatrix} \bar{L}_1(\chi, -\chi) \\ L_1(\chi, \chi) \end{pmatrix} \tag{50}$$

and

$$\bar{L}_1(\chi, -\chi) = \mathrm{i}c_1 \qquad L_1(\chi, -\chi) = 0 \tag{51}$$

we find

$$\bar{L}_1(\chi, \chi) = \frac{\mathrm{i}c_1 k \mathrm{e}^{-2\mathrm{i}\xi_1\chi}}{k\cosh(2k\chi) - \eta_1 \sinh(2k\chi)} \tag{52}$$

$$L_1(\chi, \chi) = -\frac{c_1 c_1^* \sinh(2k\chi)}{k\cosh(2k\chi) - \eta_1 \sinh(2k\chi)}. \tag{53}$$

Thus the potential $q(\chi) \equiv 2\bar{L}_1(\chi, \chi)$ is exponentially decaying if $\eta_1^2 > c_1 c_1^*$ and periodic (and also singular!) if $\eta_1^2 < c_1 c_1^*$. This latter case points out that if one would indeed ever have $\eta_1^2 < c_1 c_1^*$, then the virtual soliton could possibly propagate out to large χ, and also become essentially singular. However, as we shall see, for our example, this never occurs. For the case when $\eta_1^2 = c_1 c_1^*$, we simply take the limit of $k \to 0$, obtaining

$$q_2(\chi) = \bar{L}_1(\chi, \chi) = \frac{\mathrm{i}c_1 \mathrm{e}^{-2\mathrm{i}\xi_1\chi}}{1 - 2\eta_1\chi}. \tag{54}$$

Note that in the above, $\eta_1 < 0$. Thus the one-virtual-soliton potential given by (54), is nonsingular for $\chi > 0$.

Now, let us return to our τ-dependent problem. From (43) and (44), we obtain the limiting form of ζ_1 and c_1. In this limit, we have that $k \to 0$, and from (54), and to the lowest order in τ, we find that

$$q_2^{(1)} \to \frac{\tau}{1+2\chi\tau}. \tag{55}$$

Given (55), it then becomes a simple matter to solve (11) for an exact real solution for q_1, which is

$$q_1^{(1)} \to \frac{1}{1+2\chi\tau}. \tag{56}$$

This one-virtual-soliton potential, in the limit of small τ, is exactly the real amplitude solution considered in [10]. Thus we have that the solution generated by this summation over the poles in the lower half plane, has the correct limit in the limit of vanishing detuning, $\Delta \to 0$.

6. The N-pole potentials

Now, we need the solution in the limit of $N \to \infty$. To this end, let us construct the general N-pole potential, with the intention of eventually taking this limit. We define

$$\begin{aligned} h_j(\chi, y) &\equiv \mathrm{e}^{-\mathrm{i}\zeta_j y}\left(\int_{-y}^{\chi} L_1(\chi, s)\mathrm{e}^{-\mathrm{i}\zeta_j s}\,\mathrm{d}s - \mathrm{e}^{-\mathrm{i}\zeta_j \chi}\right) \\ \bar{h}_j(\chi, y) &\equiv \mathrm{e}^{-\mathrm{i}\zeta_j^* y}\int_{y}^{\chi} \bar{L}_1(\chi, s)\mathrm{e}^{\mathrm{i}\zeta_j^* s}\,\mathrm{d}s \end{aligned} \tag{57}$$

for $j = 1, 2, \ldots, N$. Then we get

$$(\partial_y + \mathrm{i}\zeta_j)h_j(\chi, y) = L_1(\chi, -y) \qquad (\partial_y + \mathrm{i}\zeta_j^*)\bar{h}_j(\chi, y) = -\bar{L}_1(\chi, y) \tag{58}$$

and this combination, with (42), leads to a system of $2N$ homogeneous linear ODEs with constant coefficients (χ is a fixed parameter),

$$\mathrm{i}\partial_y \boldsymbol{h}(\chi, y) = M\boldsymbol{h}(\chi, y) \qquad \boldsymbol{h} \equiv (\bar{h}_j, h_j)^T \tag{59}$$

with the $2N \times 2N$ matrix

$$M = \begin{pmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{pmatrix} \tag{60}$$

which is formed from the $N \times N$ submatrices

$$(M^{11})^* = M^{22} = \mathrm{diag}[\zeta_1 \ldots \zeta_N] \tag{61}$$

$$M^{12} = (M^{21})^* = -\begin{pmatrix} c_1 & \cdots & c_N \\ \cdots & \cdots & \cdots \\ c_1 & \cdots & c_N \end{pmatrix}. \tag{62}$$

The system (59) has to be solved with the boundary conditions

$$h_j(\chi, -\chi) = -1 \qquad \bar{h}_j(\chi, \chi) = 0. \tag{63}$$

Next, we diagonalize M,

$$M = ADA^{-1} \tag{64}$$

with D being diagonal and define the matrix

$$\Gamma(\chi) \equiv A\mathrm{e}^{2\mathrm{i}D\chi}A^{-1} \tag{65}$$

which has the property

$$\boldsymbol{h}(\chi, -\chi) = \Gamma(\chi)\boldsymbol{h}(\chi, \chi). \tag{66}$$

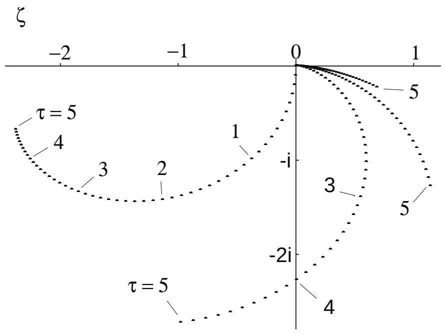

Figure 1. The first four out of an infinity of poles in the ζ-plane are depicted in steps of growing τ (retarded time).

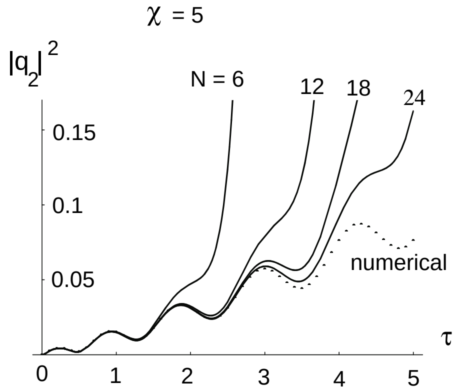

Figure 2. For $\chi = 5$ successive N-pole approximations with $N = 6, 12, 18$ and 24 are compared with the numerical solution (dotted curve). ($|q_2|^2 = |r_+|^2$ means the excitation of the medium, cf the transformation in section 2.)

Then from the boundary conditions (63) we get, in particular,

$$h(\chi, -\chi) = \Gamma^{22} h(\chi, \chi) \qquad h(\chi, \chi) = (\Gamma^{22})^{-1} h(\chi, -\chi) \tag{67}$$

where Γ^{22} is the lower right $N \times N$ submatrix of Γ and $h \equiv (h_1, \ldots h_N)^T$. Now the $h_j(\chi, \chi)$ are known as per (67), (63), and when combining this with (41), (58), (59), (62) we arrive at the formula

$$q_2(\chi) = \bar{L}_1(\chi, \chi) = -\mathrm{i} \sum_{j=1}^{N} c_j h_j(\chi, \chi). \tag{68}$$

Here the τ variable, which is fixed, has been omitted for simplicity. In the limit of $N \to \infty$, we may obtain the function $q_2(\chi, \tau)$ for the considered χ–τ region. $q_1(\chi, \tau)$ can then be found by numerical differentiation, according to the second of equations (11).

7. Convergence of the N-pole potentials and their comparison with the numerical solution

The crucial question now is whether the Goursat problem under consideration could be solved approximately by such a prescribed N-pole potential, and whether—with increasing N—the potential will eventually converge to the solution.

The poles ζ_j, determined by (34) through the zeros w_j of the function $m(w, \tau)$, see (26), are depicted in figure 1. For any fixed τ, there is an infinity of poles clustering at the origin. From the figure, one may see how the first four poles move as functions of τ. For the poles near to the origin, the residues are small and $\propto |\zeta_j|^2$. However, at the same time, their decay constants (see (58)) also become small as well. This means that for larger values of χ, more and more poles will be necessary for convergence. Therefore an N-pole potential can be expected to be valid only in a restricted χ–τ region. Figure 2 shows how the numerical solution for $\chi = 5$ (dotted curve) is approximated by successive N-pole potentials, with $N = 6, 12, 18, 24$. It is clearly seen that the approximation becomes better and better with growing N. For $N = 24$ the curves coincide for $\tau < 3$.

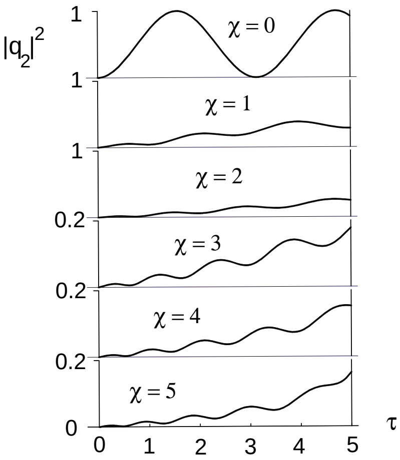

Figure 3. The excitation of the medium from the 24-pole approximation is depicted in dependence on τ for $\chi = 0$–5.

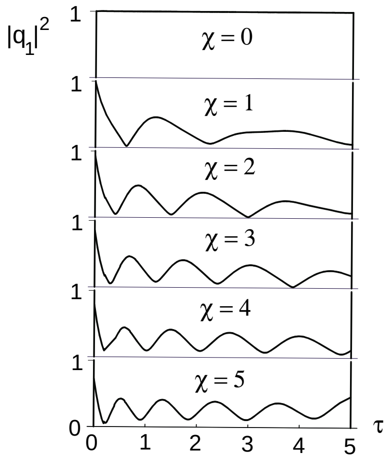

Figure 4. The output $|q_1|^2 = |\mathcal{E}|^2$ of the optical pulse is depicted for probe lengths $\chi = 1$–5 and compared with the input at $\chi = 0$. Here 24 poles are used just as in figure 3.

In figures 3 and 4 respectively, $|q_2|^2 = |r_+|^2$ (the excitation of the medium) and $|q_1|^2 = |\mathcal{E}|^2$ (the intensity of the optical pulse) are depicted as functions of the retarded time τ for $\chi = 0$–5. From the physical point of view we are mostly interested in figure 4 which directly illustrates the propagation of an optical square pulse in a two-photon absorber with detuning. There is some modulation of the pulse with the modulation time (=distance of maximum values) growing with τ and decreasing with χ while the modulation depth is slightly decreasing with both variables.

8. Summary and conclusion

In this paper, together with the preceding one [10], it is shown that the inverse scattering method can be applied to a finite or semi-infinite interval by use of the *effective S*-matrix. So far, we have studied AKNS problems and found poles in the upper or lower half $\zeta-$plane for $r = -q^*$ or $r = +q^*$ respectively. Apart from trivial examples there is an infinity of poles. For solutions in a restricted space–time region, it makes sense to construct solutions by using a finite number of poles.

The numerical effort required for establishing an N-soliton potential is twofold: first one has to solve a transcendental equation (28) to obtain the N leading poles ζ_j, see (34). Then one has to carry out the linear algebra as described in section 6.

At the moment, there is no indication that one can obtain the asymptotic form of the solutions. However, with this formulation here, it can be anticipated that in the limit of $N \to \infty$, such may possibly be gleaned.

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